Convergence Rates for Inverse Problems with Impulsive Noise

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Outline

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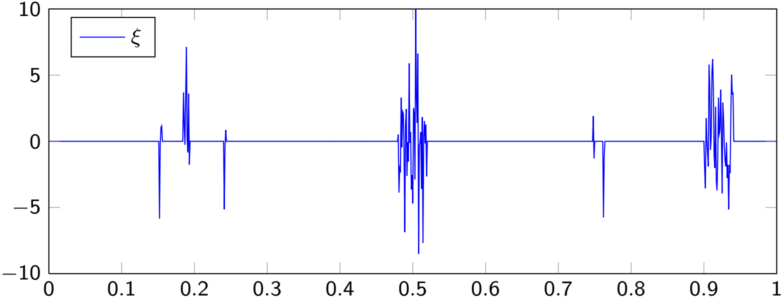
What is Impulsive Noise?

- noise ξ is small in large parts of the domain $\mathbb{M}$, but large on small parts of the domain
- occurs e.g. in digital image acquisition
- caused by faulty memory locations, malfunctioning pixels etc.
- popular example: salt-and-pepper noise



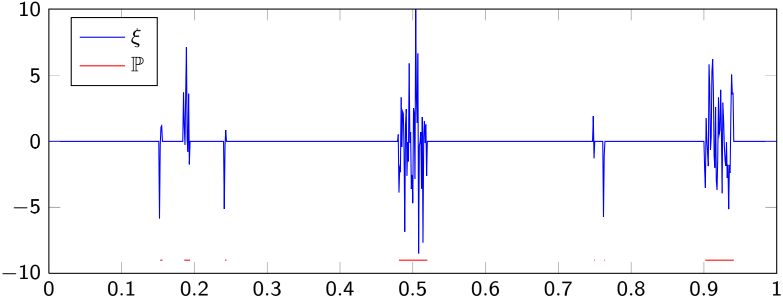
A continuous model for impulsive noise

Suppose $\xi \in \mathbf{L}^1(\mathbb{M})$, $\mathfrak{B}(\mathbb{M}) \hat{=}$ Borel σ-algebra of $\mathbb{M}$.

Noise model

There exist two parameters $\varepsilon, \eta \geq 0$ such that

$$\exists\ \mathbb{P} \in \mathfrak{B}(\mathbb{M}): \qquad \|\xi\|_{\mathbf{L}^1(\mathbb{M}\setminus\mathbb{P})} \leq \varepsilon, \qquad |\mathbb{P}| \leq \eta\,.$$



Inverse Problems with Impulsive Noise

- we want to reconstruct $f^\dagger$ from

$$g^{\mathrm{obs}} = F\left(f^\dagger\right) + \xi =: g^\dagger + \xi$$

where ξ is impulsive noise
- natural setup: $F : D(F) \subset \mathcal{X} \to \mathbf{L}^1(\mathbb{M}) \subseteq \mathcal{Y}$, possibly nonlinear
- Favorable method: Tikhonov regularization

$$\widehat{f}_\alpha \in \operatorname*{argmin}_{f \in D(F)} \left[\frac{1}{\alpha r} \left\|F(f) - g^{\mathrm{obs}}\right\|_{\mathcal{Y}}^r + \mathcal{R}(f)\right]$$

- Minimizer $\widehat{f}_\alpha$ exists under reasonable assumptions.

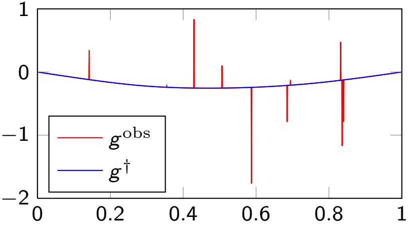
How to choose $\mathcal{Y}$ and r

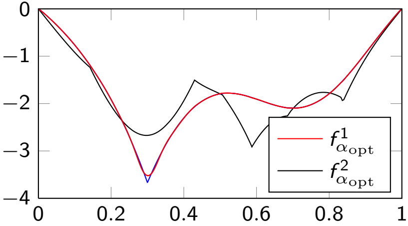
here: F = linear integral operator (two times smoothing) on $\mathbb{M} = [0, 1]$

$$f_\alpha^r = \underset{f \in \mathbf{L}^2(\mathbb{M})}{\operatorname{argmin}} \left[\frac{1}{r\alpha} \left\| F(f) - g^{\mathrm{obs}} \right\|_{\mathbf{L}^r(\mathbb{M})}^r + \|f\|_{\mathbf{L}^2(\mathbb{M})}^2 \right], \qquad r = 1, 2$$

computation of f_α^1 via dual formulation, see e.g.

C. Clason, B. Jin, K. Kunisch.
A semismooth Newton method for $\mathbf{L}^1$ data fitting with automatic choice of regularization parameters and noise calibration.
SIAM J. Imaging Sci., 3:199–231, 2010.





Outline

Theoretical state of the art

- known theory provides rates of convergence as $\|\xi\|_{\mathcal{Y}}$ tends to 0
- this does not fully explain the remarkable quality of the $\mathbf{L}^1$-reconstruction!

Example: 'Most impulsive' noise. $\mathcal{Y} = \mathfrak{M}(\mathbb{M})$ (space of all signed measures) and

$$\xi = \sum_{j=1}^{N} c_j \delta_{x_j}$$

with $N \in \mathbb{N}$, $c_j \in \mathbb{R}$ and $x_j \in \mathbb{M}$ for $1 \leq j \leq N$.

Then $\|\xi\|_{\mathfrak{M}(\mathbb{M})} = \sum_{j=1}^{N} |c_j|$ might be large! However

$$\left\| g - g^{\text{obs}} \right\|_{\mathfrak{M}(\mathbb{M})} = \left\| g - g^{\dagger} \right\|_{\mathbf{L}^1(\mathbb{M})} + \sum_{j=1}^{N} |c_j| = \left\| g - g^{\dagger} \right\|_{\mathbf{L}^1(\mathbb{M})} + \|\xi\|_{\mathfrak{M}(\mathbb{M})}.$$

So ξ does not influence the minimizer $\widehat{f}_\alpha$!

Improving the noise level

'Most impulsive' noise ξ influences $g \mapsto \left\|g - g^{\mathrm{obs}}\right\|_{\mathfrak{M}(\mathbb{M})}$ only as an additive constant, no influence on $\widehat{f}_\alpha$!
Idea: For general ξ study the influence of ξ on the data fidelity term $\left\|g - g^{\mathrm{obs}}\right\|_{\mathcal{Y}}^r$ for all g.

Variational noise assumption

Suppose there exist $C_{\mathrm{err}} > 0$ and a noise level function **err** : $F(D(F)) \to [0, \infty]$ such that

$$\left\|g - g^{\mathrm{obs}}\right\|_{\mathcal{Y}}^r - \|\xi\|_{\mathcal{Y}}^r \geq \frac{1}{C_{\mathrm{err}}} \left\|g - g^{\dagger}\right\|_{\mathcal{Y}}^r - \mathbf{err}(g), \qquad g \in F(D(F)).$$

Examples for the noise function **err**

$$\left\|g - g^{\mathrm{obs}}\right\|_{\mathcal{Y}}^{r} - \|\xi\|_{\mathcal{Y}}^{r} \geq \frac{1}{C_{\mathrm{err}}} \left\|g - g^{\dagger}\right\|_{\mathcal{Y}}^{r} - \mathbf{err}\,(g)\,, \qquad g \in F(D\,(F)).$$

1. It follows from the triangle inequality that the Assumption is always fulfilled with

$$C_{\mathrm{err}} = 2^{r-1} \qquad \text{and} \qquad \mathbf{err} \equiv 2\,\|\xi\|_{\mathcal{Y}}^{r}\,.$$

2. In the Example of 'most impulsive' noise ($\mathcal{Y} = \mathfrak{M}\,(\mathbb{M})$, $r = 1$) the Assumption holds true with the optimal parameters

$$C_{\mathrm{err}} = 1 \qquad \text{and} \qquad \mathbf{err} \equiv 0.$$

Convergence analysis under the variational noise assumption

- Bregman distance:

$$\mathcal{D}_{\mathcal{R}}^{f^*}\left(f, f^\dagger\right) := \mathcal{R}\left(f\right) - \mathcal{R}\left(f^\dagger\right) - \left\langle f^*, f - f^\dagger\right\rangle$$

  where $f^* \in \partial\mathcal{R}\left(f^\dagger\right) \subset \mathcal{X}'$.

- use a variational inequality as source condition:

$$\beta\mathcal{D}_{\mathcal{R}}^{f^*}\left(f, f^\dagger\right) \leq \mathcal{R}\left(f\right) - \mathcal{R}\left(f^\dagger\right) + \varphi\left(\left\|F\left(f\right) - g^\dagger\right\|_{\mathcal{Y}}^{r}\right)$$

  for all $f \in D\left(F\right)$ with $\beta > 0$. φ is assumed to fulfill
  - $\varphi\left(0\right) = 0$,
  - $\varphi \nearrow$,
  - φ concave.

Convergence rates

suppose

- the noise assumption is fulfilled with a function **err** ≥ 0 and
- the variational inequality holds true.

Theorem (error decomposition)

$$\beta \mathcal{D}_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right) \leq \frac{\mathbf{err}\left(F\left(\widehat{f}_\alpha\right)\right)}{r\alpha} + (-\varphi)^*\left(-\frac{1}{rC_{\mathrm{err}}\alpha}\right),$$

$$\left\|F\left(\widehat{f}_\alpha\right) - g^\dagger\right\|_{\mathcal{Y}}^r \leq \frac{C_{\mathrm{err}}}{\lambda}\mathbf{err}\left(F\left(\widehat{f}_\alpha\right)\right) + \frac{rC_{\mathrm{err}}\alpha}{\lambda}(-\varphi)^*\left(-\frac{1-\lambda}{rC_{\mathrm{err}}\alpha}\right)$$

for all $\alpha > 0$ and $\lambda \in (0,1)$.

Fenchel conjugate:

$$(-\varphi)^*(s) = \sup_{\tau \geq 0}\left(s\tau + \varphi(\tau)\right).$$

Outline

Working schedule

- consider Tikhonov regularization for Inverse Problems with Impulsive Noise ($\mathcal{Y} = \mathbf{L}^1(\mathbb{M})$, $r = 1$):

$$\widehat{f}_\alpha \in \underset{f \in D(F)}{\operatorname{argmin}} \left[\frac{1}{\alpha} \left\| F(f) - g^{\mathrm{obs}} \right\|_{\mathbf{L}^1(\mathbb{M})} + \mathcal{R}(f) \right]$$

- recall: noise ξ fulfills

$$\exists\, \mathbb{P} \in \mathfrak{B}(\mathbb{M}) : \qquad \|\xi\|_{\mathbf{L}^1(\mathbb{M} \setminus \mathbb{P})} \leq \varepsilon, \qquad |\mathbb{P}| \leq \eta$$

$\rightsquigarrow$ need to estimate $\mathbf{err}(g)$ with $g = F\left(\widehat{f}_\alpha\right)$ defined by

$$\left\| g - g^{\mathrm{obs}} \right\|_{\mathbf{L}^1(\mathbb{M})} - \|\xi\|_{\mathbf{L}^1(\mathbb{M})} \geq \frac{1}{C_{\mathrm{err}}} \left\| g - g^{\dagger} \right\|_{\mathbf{L}^1(\mathbb{M})} - \mathbf{err}(g)$$

First step: triangle inequalities

$$\left\|g - g^{\mathrm{obs}}\right\|_{\mathbf{L}^1(\mathbb{M})} - \|\xi\|_{\mathbf{L}^1(\mathbb{M})} \geq \frac{1}{C_{\mathrm{err}}} \left\|g - g^{\dagger}\right\|_{\mathbf{L}^1(\mathbb{M})} - \mathbf{err}\,(g)$$

$$\begin{aligned}\left\|g - g^{\mathrm{obs}}\right\|_{\mathbf{L}^1(\mathbb{M})} - \|\xi\|_{\mathbf{L}^1(\mathbb{M})} &= \int\limits_{\mathbb{M}\setminus\mathbb{P}} \left[\left|g^{\mathrm{obs}} - g\right| - |\xi|\right]\,\mathrm{d}x + \int\limits_{\mathbb{P}} \left[\left|g^{\mathrm{obs}} - g\right| - |\xi|\right]\,\mathrm{d}x \\ &\geq \left\|g - g^{\dagger}\right\|_{\mathbf{L}^1(\mathbb{M}\setminus\mathbb{P})} - 2\varepsilon - |\mathbb{P}|\left\|g - g^{\dagger}\right\|_{\mathbf{L}^\infty(\mathbb{P})} \\ &\geq \left\|g - g^{\dagger}\right\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2\eta\left\|g - g^{\dagger}\right\|_{\mathbf{L}^\infty(\mathbb{P})}\end{aligned}$$

Here we used

- the first triangle inequality on $\mathbb{M} \setminus \mathbb{P}$ and
- the second triangle inequality on $\mathbb{P}$.

Second step: improving the bound

$$\left\|g - g^{\mathrm{obs}}\right\|_{\mathbf{L}^1(\mathbb{M})} - \|\xi\|_{\mathbf{L}^1(\mathbb{M})} \geq \left\|g - g^{\dagger}\right\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2\eta \left\|g - g^{\dagger}\right\|_{\mathbf{L}^\infty(\mathbb{P})}$$

If F is smoothing and $g = F(f)$, then $\left\|g - g^{\dagger}\right\|_{\mathbf{L}^\infty(\mathbb{P})}$ also decays with η!

Theorem (Hohage, W.)

If $k > d/p$, then for all $C_{\mathrm{err}} > 1$ there exist $C > 0$ and $\eta_0 > 0$ such that

$$\|v\|_{\mathbf{L}^\infty(\mathbb{M})} \leq C\eta^{\frac{k}{d}-\frac{1}{p}} |v|_{W^{k,p}(\mathbb{M})} + \frac{C_{\mathrm{err}} - 1}{2C_{\mathrm{err}}\eta} \|v\|_{\mathbf{L}^1(\mathbb{M})}$$

for all $v \in W^{k,p}(\mathbb{M})$ and $\eta \in (0, \eta_0]$.

Follows from techniques used in approximation theory / FEM analysis (Ehrling's lemma and Sobolev's embedding theorem).

Second step: improving the bound (cont')

Smoothing assumption on F

$\mathbb{M} \subset \mathbb{R}^d$ bounded & Lipschitz, $\exists\ k \in \mathbb{N}_0, p \in [1, \infty], k > d/p$ and $q \in (1, \infty)$ such that

$$F(D(F)) \subset W^{k,p}(\mathbb{M}) \quad \text{and} \quad \left| F(f) - g^\dagger \right|_{W^{k,p}(\mathbb{M})} \leq C_{F,k,p} \mathcal{D}_{\mathcal{R}}^{f^*}\left(f, f^\dagger\right)^{\frac{1}{q}}$$

for all $f \in D(F)$ with some $C_{F,k,p} > 0$.

This allows us to use $v = F(f) - g^\dagger$, e.g. it follows

$$\left\| F(f) - g^\dagger \right\|_{\mathbf{L}^\infty(\mathbb{M})} \leq C\eta^{\frac{k}{d} - \frac{1}{p}} \left| F(f) - g^\dagger \right|_{W^{k,p}(\mathbb{M})} + \frac{C_{\mathrm{err}} - 1}{2C_{\mathrm{err}}\eta} \left\| F(f) - g^\dagger \right\|_{\mathbf{L}^1(\mathbb{M})}$$

whenever η is sufficiently small.

Second step: improving the bound (cont')

$$\begin{aligned}
&\left\| F(f) - g^{\mathrm{obs}} \right\|_{\mathbf{L}^1(\mathbb{M})} - \|\xi\|_{\mathbf{L}^1(\mathbb{M})} \\
\geq &\left\| F(f) - g^{\dagger} \right\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2\eta \left\| F(f) - g^{\dagger} \right\|_{\mathbf{L}^\infty(\mathbb{P})} \\
\geq &\left(1 - \frac{C_{\mathrm{err}} - 1}{C_{\mathrm{err}}}\right) \left\| F(f) - g^{\dagger} \right\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2C\eta^{\frac{k}{d} - \frac{1}{p} + 1} \left| F(f) - g^{\dagger} \right|_{W^{k,p}(\mathbb{M})} \\
\geq &\frac{1}{C_{\mathrm{err}}} \left\| F(f) - g^{\dagger} \right\|_{\mathbf{L}^1(\mathbb{M})} - 2\varepsilon - 2CC_{F,k,p}\eta^{\frac{k}{d} - \frac{1}{p} + 1} \mathcal{D}_{\mathcal{R}}^{f^*}\left(f, f^{\dagger}\right)^{\frac{1}{q}} \\
\overset{!}{\geq} &\frac{1}{C_{\mathrm{err}}} \left\| F(f) - g^{\dagger} \right\|_{\mathbf{L}^1(\mathbb{M})} - \mathbf{err}\left(F(f)\right)
\end{aligned}$$

$$\left\| F(f) - g^{\dagger} \right\|_{\mathbf{L}^\infty(\mathbb{M})} \leq C\eta^{\frac{k}{d} - \frac{1}{p}} \left| F(f) - g^{\dagger} \right|_{W^{k,p}(\mathbb{M})} + \frac{C_{\mathrm{err}} - 1}{2C_{\mathrm{err}}\eta} \left\| F(f) - g^{\dagger} \right\|_{\mathbf{L}^1(\mathbb{M})}$$

$$\left| F(f) - g^{\dagger} \right|_{W^{k,p}(\mathbb{M})} \leq C_{F,k,p} \mathcal{D}_{\mathcal{R}}^{f^*}\left(f, f^{\dagger}\right)^{\frac{1}{q}}$$

Thus for any $C_{\mathrm{err}} > 1$ we can choose

$$\mathbf{err}\left(F(f)\right) = 2\varepsilon + 2C_{F,k,p} C\eta^{\frac{k}{d} - \frac{1}{p} + 1} \mathcal{D}_{\mathcal{R}}^{f^*}\left(f, f^{\dagger}\right)^{\frac{1}{q}}$$

Third step: final estimate for $\mathbf{err}\left(F\left(\widehat{f}_\alpha\right)\right)$

Calculation above:

$$\mathbf{err}\left(F\left(\widehat{f}_\alpha\right)\right) = 2\varepsilon + 2C_{F,k,p}C\eta^{\frac{k}{d}-\frac{1}{p}+1}\mathcal{D}_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right)^{\frac{1}{q}}$$

General convergence analysis:

$$\beta\mathcal{D}_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right) \leq \frac{\mathbf{err}\left(F\left(\widehat{f}_\alpha\right)\right)}{\alpha} + (-\varphi)^*\left(-\frac{1}{C_{\mathrm{err}}\alpha}\right)$$

This implies using Young's inequality and $(a+b)^{\frac{1}{q}} \leq a^{\frac{1}{q}} + b^{\frac{1}{q}}$ that

$$\mathbf{err}\left(F\left(\widehat{f}_\alpha\right)\right) \leq 2q'\varepsilon + (q'-1)\frac{\eta^{\frac{q'k}{d}+\frac{q'(p-1)}{p}}}{\alpha^{q'-1}} + C'(-\varphi)^*\left(-\frac{1}{C_{\mathrm{err}}\alpha}\right)$$

where $1/q + 1/q' = 1$ and $C' > 0$ whenever $\alpha > 0$ and $\eta \geq 0$ is sufficiently small.

Error bound for Tikhonov regularization

Insert the estimate for **err** $\left(F\left(\widehat{f}_\alpha\right)\right)$ into the general error decomposition to obtain

Theorem (Hohage, W.)

Suppose the variational inequality is fulfilled and F obeys the smoothing assumption. Then we have for arbitrary $C_{\mathrm{err}} > 1$ and all $\alpha > 0$ and $\eta > 0$ sufficiently small

$$\beta \mathcal{D}_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right) \leq 2q' \frac{\varepsilon}{\alpha} + (q'-1)\frac{\eta^{\frac{q'k}{d}+\frac{q'(p-1)}{p}}}{\alpha^{q'}} + C'(-\varphi)^*\left(-\frac{1}{C_{\mathrm{err}}\alpha}\right)$$

$$\left\|F\left(\widehat{f}_\alpha\right) - g^\dagger\right\|_{\mathbf{L}^1(\mathbb{M})} \leq 4q'\varepsilon + 2(q'-1)\frac{\eta^{\frac{q'k}{d}+\frac{q'(p-1)}{p}}}{\alpha^{q'-1}} + 2C'C_{\mathrm{err}}\alpha(-\varphi)^*\left(-\frac{1}{C_{\mathrm{err}}\alpha}\right)$$

For simplicity we study only $q = 2$ and $\varphi(\tau) = c\tau^\kappa$ with $c > 0$ and $\kappa \in (0,1)$ in the following.

An optimal a priori parameter choice

$$\beta \mathcal{D}_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right) \leq 4\frac{\varepsilon}{\alpha} + \frac{\eta^{\frac{2k}{d}+\frac{2(p-1)}{p}}}{\alpha^2} + C'\left(-\varphi\right)^*\left(-\frac{1}{C_{\mathrm{err}}\alpha}\right)$$

If $\varphi(t) = c \cdot t^\kappa$ with $c > 0$ and $\kappa \in (0,1)$, then $(-\varphi)^*\left(-\frac{1}{\alpha}\right) = C \cdot \alpha^{\frac{\kappa}{1-\kappa}}$.
So for $\alpha \sim \max\left\{\varepsilon^{1-\kappa}, \eta^{\left(\frac{1-\kappa}{2-\kappa}\right)\left(\frac{2k}{d}+\frac{2(p-1)}{p}\right)}\right\}$ we obtain

$$\mathcal{D}_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right) = \mathcal{O}\left(\max\left\{\varepsilon^\kappa, \eta^{\frac{\kappa\gamma}{2-\kappa}}\right\}\right)$$

with $\gamma := \frac{2k}{d} + \frac{2(p-1)}{p}$ as $\max\{\varepsilon, \eta\} \searrow 0$.

Functional dependence of ε and η

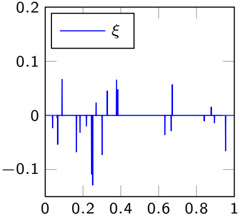
$$\exists\ \mathbb{P} \in \mathfrak{B}(\mathbb{M}): \qquad \|\xi\|_{\mathbf{L}^1(\mathbb{M}\setminus\mathbb{P})} \leq \varepsilon, \qquad |\mathbb{P}| \leq \eta \tag{1}$$

- model allows for different choices of ε and η which depend on each other
- study the dependence function

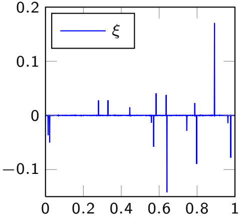
$$\varepsilon_\xi(\eta) := \inf\left\{ \|\xi\|_{\mathbf{L}^1(\mathbb{M}\setminus\mathbb{P})} \;\middle|\; \mathbb{P} \in \mathfrak{B}(\mathbb{M}), |\mathbb{P}| \leq \eta \right\}.$$

- then for any $\eta \geq 0$ eq. (1) is fulfilled with $\varepsilon = \varepsilon_\xi(\eta)$
- for $\xi \in \mathbf{L}^1(\mathbb{M})$ the following holds true:
  1. $\varepsilon_\xi(0) = \|\xi\|_{\mathbf{L}^1(\mathbb{M})}$, $\varepsilon_\xi(|\mathbb{M}|) = 0$
  2. ε_ξ is continuous, decreasing, and convex

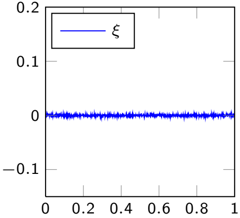
Examples for ε_ξ



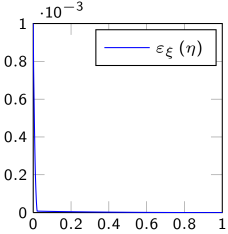
Impulsive noise

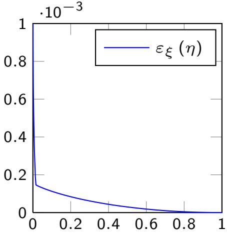


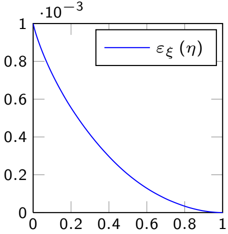
Impulsive noise?



Gaussian white noise







Convergence rates in terms of an optimal η

- Recall: $\mathcal{D}_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right) = \mathcal{O}\left(\max\left\{\varepsilon^\kappa, \eta^{\frac{\kappa\gamma}{2-\kappa}}\right\}\right)$
- Substituting ε by $\varepsilon_\xi(\eta)$ yields

$$\mathcal{D}_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right) \leq C \inf_{0\leq\eta\leq|\mathbb{M}|}\left[\varepsilon_\xi(\eta)^\kappa + \eta^{\frac{\kappa}{2-\kappa}\gamma}\right] \qquad \text{as} \qquad \xi \to 0$$

- Note that ξ and ε_ξ are unknown in general, but possibly an upper bound for ε_ξ can be calculated
- As $\varepsilon_\xi \searrow$ and $\eta^{\frac{\kappa}{2-\kappa}\gamma} \nearrow$ in η, there exists an intersecting point $\bar{\eta} > 0$
- Thus we have

$$\mathcal{D}_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right) \leq 2C\varepsilon_\xi(\bar{\eta})^\kappa \qquad \text{as} \qquad \xi \to 0$$

- The state-of-the-art analysis yields $(\eta = 0)$

$$\mathcal{D}_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right) \leq \tilde{C}\varepsilon_\xi(0)^\kappa \qquad \text{as} \qquad \xi \to 0.$$

$\rightsquigarrow$ improvement measured by the factor $\left(\varepsilon_\xi(0)/\varepsilon_\xi(\bar{\eta})\right)^\kappa$, which is arbitrary large for impulsive noise

Outline

Considered operator

- $\mathbb{M} = [0,1]$ and $T : \mathbf{L}^2(\mathbb{M}) \to \mathbf{L}^2(\mathbb{M})$ defined by

$$(Tf)(x) = \int_0^1 k(x,y) f(y)\,\mathrm{d}y, \qquad x \in \mathbb{M}$$

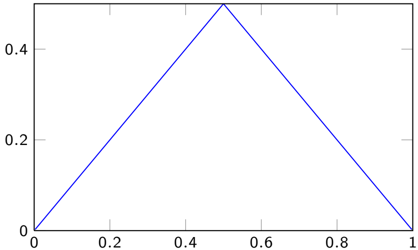
  with kernel $k(x,y) = \min\{x \cdot (1-y), y \cdot (1-x)\}, x, y \in \mathbb{M}$.
- then $(Tf)'' = f$ for any $f \in \mathbf{L}^2(\mathbb{M})$ and T is 2 times smoothing ($k = 2$ and $p = 2$).
- the smoothing Assumption is valid with exponent $\gamma = 2k/d + 2(p-1)/p = 5$ and $q = 2$
- discretization: equidistant points $x_1 = \frac{1}{2n}, x_2 = \frac{3}{2n}, \dots, x_n = \frac{2n-1}{2n}$ and composite midpoint rule

$$(Tf)(x) = \int_0^1 k(x,y) f(y)\,\mathrm{d}y \approx \frac{1}{n} \sum_{i=1}^{n} k(x, x_i) f(x_i).$$

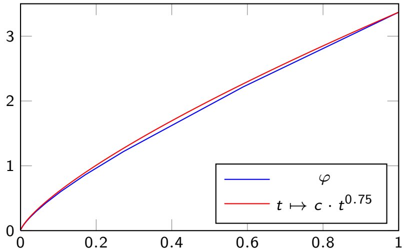
Simulations

- $f^\dagger$ and $g^\dagger$ are calculated analytically to avoid an inverse crime
- we consider 'purely impulsive noise' ($\varepsilon = 0$) for different values of η
- generation of ξ:
    - given η, choose randomly $\lceil \eta \cdot n \rceil$ grid points forming $\mathbb{P}$
    - simulate ξ such that $\xi_{|_{\mathbb{M}\setminus\mathbb{P}}} = 0$ and $\xi_{|_{\mathbb{P}}} = \pm 1/\eta$ with probability 1/2 respectively for each $x_i \in \mathbb{P}$
- for each $\eta_j = (4/5)^j$, $j = 1, ...$ we perform 10 experiments
- in each experiment α is chosen optimally by trial and error
- following plots show η vs. empirical mean of $\mathcal{D}_{\mathcal{R}}^{f^*}\left(\widehat{f}_\alpha, f^\dagger\right)$

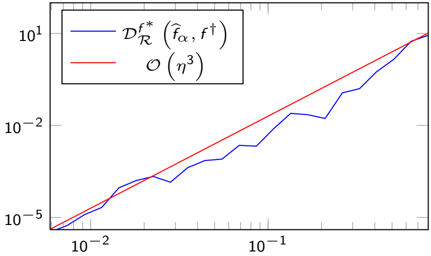
Example 1



(a) Exact solution $f^\dagger$

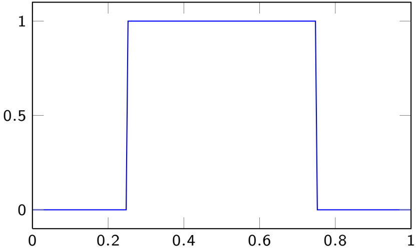


(b) Estimated index function φ

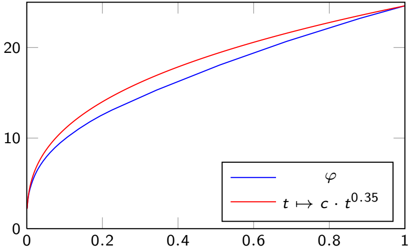


(c) mean convergence in $\mathcal{X}$

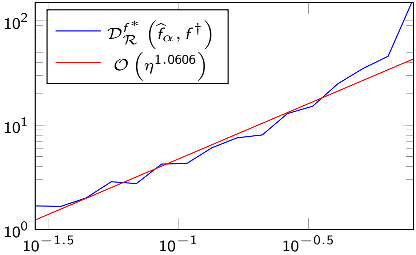
Example 2



(d) Exact solution $f^\dagger$



(e) Estimated index function φ



(f) mean convergence in $\mathcal{X}$

Outline

Presented results and future work

- Inverse Problems with Impulsive noise
  - continuous model for Impulsive noise
  - improved convergence rates
- numerical examples suggest order optimality
- future work: infinitely smoothing operators!

T. Hohage and F. Werner
Convergence rates for Inverse Problems with Impulsive Noise.
Submitted, *arXiv:* 1308.2536.

Thank you for your attention!